5. Sedov, L. I., Similarity and Dimensional Methods in Mechanics, 4th ed. Moscow, Gostekhizdat, 1957.
6. Korobeinikov, V. P., Mel'nikov, N. S. and Riazanov, E. V., Theory of Point Explosions. Moscow, Fizmatgiz, 1961.
7. Staniukovich, K. P., Unsteady Motions of Contiuous Media. Moscow, Gostekhizdat, 1955.
8. Blythe, P. A., Near-frozen quasi-one-dimensional flow. Phil. Trans. Roy. Soc. (London) Series A, Vol. 262, №1125, 1967.
9. Ryzhov, O. S. and Taganov, G. I., The second limiting case of the intense explosion problem. PMM Vol. 20, №4, 1956.
10. Korobeinikov, V. P., The problem of an intense point explosion in a gas with a zero temperature gradient. Dokl. Akad. Nauk SSSR Vol. 109, №2, 1956.
11. Sidorkina, S. I., On certain motions of an aerosol. Dokl. Akad. Nauk SSSR Vol. 112, №3, 1957.
12. Bechert, K., Differentialgleichungen der Wellenausbreitung in Gasen. Ann. Physik Vol. 39, №5, p. 357, 1941.
13. Sedov, L. I., On certain unsteady motions of a compressible fluid. PMM Vol. 9, №4, 1945.
14. Staniukovich, K. P., Self-similar solutions of centrally symmetric hydrodynamic equations. Dokl. Akad. Nauk SSSR, Vol. 48, №5, 1945.
15. Chernyi, G. G., High Supersonic Velocity Gas Flows. Moscow, Fizmatgiz, 1959.
16. Brushlinskii, K. V. and Kazhdan, Ia. M., On the self-similar solutions of certain problems of gas dynamics. Uspekhi Mat. Nauk Vol. 18, №2, 1963.
17. Bishimov, E., Korobeinikov, V. P., Levin, V. A. and Chernyi, G. G., One-dimensional unsteady flows of a fuel gas mixture with allowance for the finite rate of chemical reactions. Izv. Akad. Nauk SSSR Mekh. Zhidkostei i Gazov №6, 1968.
18. Zel'dovich, Ia. B. and Kompaneets, A. S., Theory of Detonation. Moscow, Gostekhizdat, 1955.

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ON THE DIFFRACTION OF SHOCK WAVES

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It is shown that for sufficiently large values of the adiabatic exponent of a gas the solution of the problem of diffraction of an arbitrarily intense shock wave at a small angle is of a form different from the usual one, and that the solution of this problem takes three forms in the general case. The corresponding pressure formulas for the case in question are derived.

The problem of diffraction of an arbitrarily intense shock wave at a small angle was first investigated by Lighthill [1], who, however, did not go so far as to obtain complete analytic solution. This was one of the factors which led Ting and Ludloff [2] to reconsider the problem using a different method of solution. These authors succeeded in obtaining analytic formulas for the pressure. According to Ludloff [3], the formula for the pressure in the case of supersonic flows behind the advancing shock is a sum of four terms containing arc cosines. In the case of subsonic flows behind the advancing shock two of the terms containing arc cosines must be replaced by suitable terms containing hyperbolic arc cosines. From the standpoint of formula structure, this would imply the existence of just two forms of the solution of the problem in the general case.

It was subsequently shown in [4] that the Lighthill method can also be used to generalize the problem all the way to complete analytic formulas. This was demonstrated by deriving both forms of the solution for the pressure at the diffracted surface mentioned in [3].

Further analysis using the Lighthill method showed, however, that the two forms in question are generally applicable only if the values of the adiabatic exponent are not too large. We propose to show that there exists an entire domain in the plane $M_1\gamma$ (where γ is the adiabatic exponent of the gas and M_1 is the Mach number behind the advancing shock) where the solution of the problem assumes a third form different from those noted in [3].

1. Let a plane shock wave of arbitrary intensity propagate at the velocity U in a quiescent medium bounded by a wall which veers off at a small angle δ at some point (Fig. 1). On reaching the vertex of this angle, the shock wave experiences diffraction with the formation of an unsteady flow zone (Mach reflection occurs if $\delta < 0$).

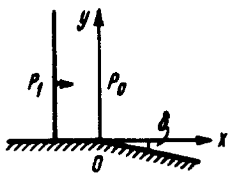

Fig. 1

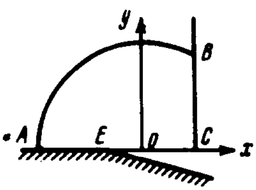

Fig. 2

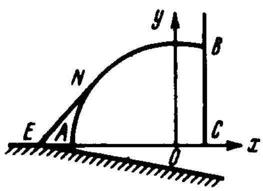

Fig. 3

In the case of subsonic flow behind the advancing shock the unsteady flow zone is of the form shown in Fig. 2; The form of this zone for a supesonic flow is shown on Fig. 3. The self-similar coordinates x, y are related to X, Y by the equations

$$x = (X - u_1 t) / a_1 t, \qquad y = Y / a_1 t$$

where u_1 and a_1 are the velocities of gas and sound, respectively, behind the advancing shock, and where the time t is measured from the instant of arrival of the shock at the vertex.

The domain $ABCA$ is defined by the relations

$$\begin{matrix} x^2 + y^2 < 1, \\ y > 0, x < k, \end{matrix} \qquad k = \left[\frac{(\gamma - 1) M^2 + 2}{2\gamma M^2 - (\gamma - 1)}\right]^{1/2} \qquad \left(M = \frac{U}{a_0}\right) \tag{1.1}$$

Here M is the Mach number of the shock advancing on the angle; $ENAE$ in Fig. 3 is the zone of stable Prandtl-Mayer flow.

The successive transformations

$$1)\ x = r\cos\theta, \quad y = r\sin\theta, \qquad 2)\ \theta_1 = \theta, \quad \rho = [1 - (1 - r^2)^{1/2}]\, r^{-1}$$

$$3)\quad z_1 = (k + ik')\{i - [2k'(\zeta - k - ik')^{-1}]\}, \quad k' = \sqrt{1 - k^2}, \quad \zeta = \rho e^{i\theta_1}$$

map the domain $ABCA$ into the upper half-plane of the complex plane z_1. In the plane z_1 the shock BC corresponds to the segment $z_1 > 1$ of the real axis; the segment CA corresponds to the real-axis segment $-1 < z_1 < 1$; the circular arc AB corresponds to the real-axis segment $z_1 < -1$. The vertex of the angle has the coordinate

$$x_0 = -\frac{(M_1+k)^2+(M_1{}^2-1)(1-k^2)}{(M_1k+1)^2}$$

Linearization of the corresponding equations of motion reduces the boundary value problem under consideration to the analysis of the quantity

$$p = \frac{p_2-p_1}{\rho_1 u_1 a_1}$$

which satisfies the differential equations

for $-1 < x_1 < 1$, $y_1 = 0$ (at the wall), and

$$\frac{\partial p}{\partial x_1} = -\frac{\varepsilon\delta\,[D(x_1-x_0)-1]}{(x_1-x_0)(1-x_1{}^2)^{1/2}[\alpha+(1-x_1)^{1/2}][\beta+(1-x_1)^{1/2}]} = F_1(x_1) \qquad (1.2)$$

for $1 < x_1 < \infty$, $y_1 = 0$ (at the shock BC)

$$\frac{\partial p}{\partial x_1} = \frac{\varepsilon\delta\,(\alpha+\beta)[D(x_1-x_0)-1]}{(x_1-x_0)(\alpha^2-1+x_1)(\beta^2-1+x_1)(x_1+1)^{1/2}} = F_2(x_1) \qquad (1.3)$$

Here p_2 is the pressure in the unsteady flow zone and ρ_1 is the density behind the advancing shock;

$$\alpha = \sqrt{2}M\,(Mk+\sqrt{M^2k^2-1}), \qquad \beta = \sqrt{2}M\,(Mk+\sqrt{M^2k^2-1})^{-1} \qquad (1.4)$$

$$D = \left[\frac{\alpha+\beta+\mu}{\alpha\beta\mu} + \frac{(M_1k+1)^2}{2BM_1(M_1+k)}\right]\left[\frac{(\alpha+\mu)(\beta+\mu)}{\alpha\beta}\right]^{-1}$$

$$\varepsilon = \frac{[\alpha+(1-x_0)^{1/2}][\beta+(1-x_0)^{1/2}]\,|1-x_0{}^2|^{1/2}}{\pi\,|1-M_1{}^2|^{1/2}}, \qquad \mu = \frac{\sqrt{2}(M_1+k)}{M_1k+1}$$

$$B = \frac{(\gamma+1)(M^2-1)}{2[(\gamma-1)M^2+2]}$$

We note that the above relations were generalized for an arbitrary γ in [4]; in [1] the quantity γ is assumed from the very beginning to have the value 1.4.

2. Expression (1.2) yields the following equation for the pressure at the wall:

$$-\frac{p(x_1)}{\varepsilon\delta} = -\frac{1}{\varepsilon\delta}\int F_1(x_1)\,dx_1 + C_1 \qquad (2.1)$$

where C_1 is an integration constant which must be determined.

Let us consider integral term (2.1). Setting $x_1 = 1-\xi^2$, we obtain

$$\frac{1}{2\varepsilon\delta}\int\frac{dx_1}{d\xi}F_1(x_1(\xi))\,d\xi = \int\frac{a_1d\xi}{(\alpha+\xi)\sqrt{2-\xi^2}} + \int\frac{a_2d\xi}{(\beta+\xi)\sqrt{2-\xi^2}} +$$

$$+\int\frac{a_3d\xi}{(\lambda+\xi)\sqrt{2-\xi^2}} + \int\frac{a_4d\xi}{(\lambda-\xi)\sqrt{2-\xi^2}} \qquad (\lambda = \sqrt{1-x_0}) \qquad (2.2)$$

$$a_1 = \frac{D(\alpha^2-\lambda^2)+1}{(\lambda^2-\alpha^2)(\alpha-\beta)}, \quad a_2 = \frac{D(\beta^2-\lambda^2)+1}{(\lambda^2-\beta^2)(\beta-\alpha)}, \quad a_3 = -\frac{1}{2\lambda(\alpha-\lambda)(\beta-\lambda)} \quad \text{(cont.)}$$

$$a_4 = -\frac{1}{2\lambda(\alpha+\lambda)(\beta+\lambda)}$$

We denote the integrals in the right side of (2.2) by $J_\nu(\xi(x_1)) = J_\nu(x_1)$ ($\nu = 1, \ldots, 4$). Expressions (2.1) and (2.2) now give us

$$\frac{1}{\varepsilon\delta}\int F_1(x_1)\,dx_1 = 2\sum_{\nu=1}^{4} J_\nu(x_1) \tag{2.3}$$

We can show that

$$J_1(x_1) = \frac{a_1}{\sqrt{\alpha^2-2}} \arcsin \frac{2+\alpha\sqrt{1-x_1}}{\sqrt{2}(\alpha+\sqrt{1-x_1})} \tag{2.4}$$

for any M_1 and γ.

Again for any γ we have (2.5)

$$J_3(x_1) = -\chi a_3 \ln \varphi_+(x_1), \qquad J_4(x_1) = \chi a_4 \ln|\varphi_-(x_1)| \quad (M_1 < 1)$$

$$J_3(x_1) = -i\chi a_3 \arcsin\theta_+(x_1), \qquad J_4(x_1) = i\chi a_4 \arcsin\theta_-(x_1) \quad (M_1 > 1)$$

(2.6)

$$\chi = \frac{1}{\sqrt{1+x_0}}, \qquad \theta_\pm(x_1) = \frac{2 \pm \sqrt{(1-x_0)(1-x_1)}}{\sqrt{2}(\sqrt{1-x_0} \pm \sqrt{1-x_1})}$$

$$\varphi_\pm(x_1) = \frac{2(2 \pm \sqrt{(1-x_0)(1-x_1)} + \sqrt{(1+x_0)(1+x_1)})}{\sqrt{1-x_1} \pm \sqrt{1-x_0}}$$

3. Let us consider the integral $J_2(x_1)$ separately. To this we make the substitution $\xi = \tau - \beta$ in $J_2(\xi)$. Then

$$a_2\int \frac{d\xi}{(\beta+\xi)\sqrt{2-\xi^2}} = a_2\int \frac{d\tau}{\tau\sqrt{2-\beta^2+2\beta\tau-\tau^2}} \tag{3.1}$$

We can show that the discriminant of the radicand in (3.1) is always negative. Moreover, (1.4) and (1.1) imply that $\beta = \beta(M, \gamma)$, so that $2 - \beta^2$ can be regarded as some function of M^2 and γ. If the plane $M^2\gamma$ contained some curve at which $2 - \beta^2 = 0$, then it would clearly be the boundary between the domains where $2 - \beta^2 < 0$ and $2 - \beta^2 > 0$. Let us show that such a curve does indeed exist.

Making use of (1.4), we can rewrite the condition $2 - \beta^2 < 0$ as

$$2Mk\sqrt{M^2k^2-1} < M^2 - 2M^2k^2 + 1 \tag{3.2}$$

We can show that the right side of (3.2) is always positive. Explicit expression (1.1) for k, enables us to rewrite this condition in the form

$$F(M^2,\gamma) = \frac{2}{\gamma-1}M^4 - \frac{3-\gamma}{\gamma-1}M^2 - 1 > 0 \tag{3.3}$$

Since $F(M^2, \gamma) = 0$ for $M^2 = 1$ and $M^2 = {}^1/_2(1-\gamma)$, and since $1 - \gamma < 0$, it follows that the function $F(M^2, \gamma)$ has a definite sign in the domain of physical values of M^2 (more precisely, in that part of the plane $M^2\gamma$ where $M^2 > 1$). In addition, since

$$\lim F(M^2,\gamma) = \infty \quad (M^2 \to \infty)$$

we infer from this that $F(M^2, \gamma) > 0$, i.e. that the right side of (3.2) is positive-definite.

The left side of (3.2) is also always positive, since, as is easy to show that $Mk > 1$. Hence, both sides of (3.2) can always be squared without violating the inequality. Condition (3.2) can then be written in the equivalent form

$$4M^4k^2 < M^4 + 2M^2 + 1 \tag{3.4}$$

Making use of explicit expression (1.1) for k, we find from (3.4) that

$$\frac{4M^4[(\gamma-1)M^2+2]}{2\gamma M^2-(\gamma-1)} < M^4 + 2M^2 + 1 \tag{3.5}$$

Since it is always the case that $2\gamma M^2 - (\gamma - 1) > 0$, condition (3.5) can be written as

$$2(2-\gamma)M^6 + (3\gamma - 7)M^4 + 2M^2 - (\gamma - 1) > 0 \tag{3.6}$$

The left side of (3.6) is a third-degree polynomial in M^2. We can show that one of its roots is equal to $(\gamma - 1)[2(2-\gamma)]^{-1}$ and the two other roots to unity. Hence, the condition $2 - \beta^2 \leqslant 0$ can now be written in its final form

$$2(2-\gamma)(M^2-1)^2\left(M^2 - \frac{\gamma-1}{2(2-\gamma)}\right) \geqslant 0 \tag{3.7}$$

The condition $2 - \beta^2 = 0$ is associated with the curves

$$\gamma = 2, \qquad M^2 = \frac{\gamma - 1}{2(2-\gamma)} = f_1(\gamma) \tag{3.8}$$

in the plane $M^2\gamma$. Figure 4 shows these curves for the range of physical values of M^2. (We note that $f_1(^5/_3) = 1$, and $f_1(2) = \infty$.)

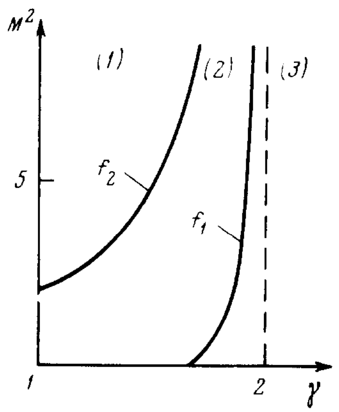

Fig. 4

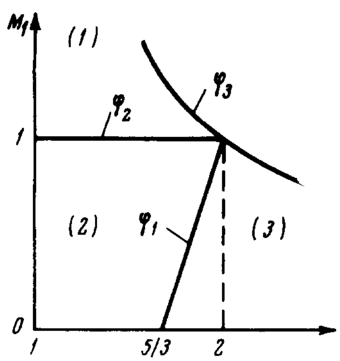

Fig. 5

Expression (3.7) implies directly that $2 - \beta^2 < 0$ to the left of the curve $f_1(\gamma)$, i.e. in domains (1) and (2). In domain (3) (except at the curve $\gamma = 2$) we always have $2 - \beta^2 > 0$.

Figure 4 also shows the curve $f_2(\gamma)$, at which $M_1 = 1$. This curve can be obtained by setting $M_1 = 1$ in the familiar expression

$$M_1 = \frac{2(M^2-1)}{\sqrt{[2\gamma M^2 - (\gamma-1)][(\gamma-1)M^2+2]}} \tag{3.9}$$

relating M_1 and M. We can show that in this case

$$M^2 = \frac{7 - \gamma + \sqrt{\gamma^2 + 2\gamma + 17}}{4(2-\gamma)} = f_2(\gamma) \quad \left(f_2(1) = \frac{3+\sqrt{5}}{2}, f_2(2) = \infty\right) \tag{3.10}$$

The curves $f_1(\gamma)$ and $f_2(\gamma)$ therefore have the same asymptote $\gamma = 2$, and since $f_2(\gamma) > f_1(\gamma)$, the foregoing implies that $M_1 > 1$ in domain (1) and $M_1 < 1$ in domains (2) and (3).

Converting from M^2, γ to the variables M_1, γ by means of transformation (3.9), we find that the indicated domains assume the form shown in Fig. 5 in the plane $M_1\gamma$ The curves $f_1(\gamma)$ and $f_2(\gamma)$ then become

$$\varphi_1(\gamma) = \frac{3\gamma - 5}{(\gamma - 1)(3 - \gamma)}, \qquad \varphi_2(\gamma) = 1 \tag{3.11}$$

respectively.

The curve $\varphi_3(\gamma)$ shown in this figure corresponds to shock waves of maximum intensity and is obtainable from (3.9) as

$$\lim_{M\to\infty} M_1 = \left(\frac{2}{\gamma(\gamma-1)}\right)^{1/2} = \varphi_3(\gamma) \tag{3.12}$$

It is clear that only the part of the plane $M_1\gamma$ which lies below this curve has physical meaning.

4. We thus have $2 - \beta^2 < 0$ in domains (1) and (2) of Fig. 5 and $2 - \beta^2 > 0$ in domain (3) (except at the curve $\gamma = 2$). It is now easy to show that

$$J_2(x_1) = \tag{4.1}$$

$$= \begin{cases} a_2(\beta^2 - 2)^{-1/2} \arcsin \{{}^1/_2 \sqrt{2}(2 + \beta\sqrt{1 - x_1})(\beta + \sqrt{1 - x_1})^{-1}\} & \text{in domains (1) and (2)} \\ -a_2(2 - \beta^2)^{-1/2} \ln \psi(x_1) & \text{in domain (3)} \end{cases}$$

$$\psi(x_1) = \frac{2(2 - \beta\sqrt{1 - x_1} + \sqrt{(2 - \beta^2)(1 + x_1)})}{\beta + \sqrt{1 - x_1}}$$

Turning now to (2.3) and recalling (2.1), we infer from (2.4)–(2.6) and (4.1) that the formula for the pressure at the wall in domain (1), i.e. for supersonic flows behind the advancing shock, is indeed (as stated in [3]) a sum of four terms containing arc cosines. As regards subsonic flows behind the advancing shock, the pressure formula is a sum of terms with two arc cosines and two hyperbolic arc cosines as stated in [3] only in domain (2).

In the subsonic cases which correspond to domain (3) of Fig. 5 the formula for the pressure (as is evident from the relations derived above) is a sum of terms containing one arc cosine and three hyperbolic arc cosines. Hence, in this case we have the third form of the solution of the problem in question which appears to have been overlooked in [3]. Let us consider this case in more detail.

Substituting the values of $J_\nu(x_1)$ for domain (3) determined from (2.1) into (2.3), we obtain the following expression for the pressure at the wall:

$$\frac{p}{2\varepsilon\delta} = \frac{a_1}{\sqrt{\alpha^2 - 2}} \arcsin \frac{2 + \alpha\sqrt{1 - x_1}}{\sqrt{2}(\alpha + \sqrt{1 - x_1})} - \frac{a_2}{\sqrt{2 - \beta^2}} \ln \psi(x_1) -$$
$$- \frac{a_3}{\sqrt{1 + x_0}} \ln \varphi_+(x_1) + \frac{a_4}{\sqrt{1 + x_0}} \ln |\varphi_-(x_1)| - \frac{C_1}{2} \tag{4.2}$$

Since the latter expression has a logarithmic singularity at the point x_0 $(-1 < x_0 < 1)$, it follows that the integration constant C_1 for the interval $-1 < x_1 < x_0$ must also be determined from the condition $p(-1) = 0$; in the interval $x_0 < x_1 < 1$ it must be determined from the condition $p(1) = p_c$, where p_c is the pressure at the point C of the shock (Figs. 2 and 3).

In order to determine p_c we must know the pressure distribution at the diffracted shock BC. We can do this by integrating (1.3),

$$p(x_1) = \int F_2(x_1)\,dx_1 + C_2 \qquad (4.3)$$

The integration constant C_2 in this expression must be determined from the self-evident condition $p(\infty) = 0$. This gives us the following formula for the pressure at the diffracted shock in domain (3):

$$-\frac{p}{k\delta} = s_1 \arccos\left(\frac{1+x_1}{\alpha^2-1+x_1}\right)^{1/2} + \frac{is_2}{2}\ln\frac{\sqrt{1+x_1}-\sqrt{2-\beta^2}}{\sqrt{1+x_1}+\sqrt{2-\beta^2}} - s_3\ln\frac{\sqrt{1+x_1}-\sqrt{1+x_0}}{\sqrt{1+x_1}+\sqrt{1+x_0}} \qquad (4.4)$$

The coefficients $s_1 - s_3$ in (4.4) are given by

$$s_1 = \frac{2a_1\varepsilon}{k\sqrt{\alpha^2-2}}, \quad s_2 = \frac{2a_2\varepsilon}{k\sqrt{\beta^2-2}}, \quad s_3 = -\frac{\varepsilon(\alpha+\beta)}{k(\alpha^2-\lambda^2)(\beta^2-\lambda^2)\sqrt{1+x_0}}$$

Using the above method to determine the integration constant C_1, we obtain the formulas for the pressure at the wall in domain (3), namely

$$-\frac{p}{k\delta} = q_1 \arccos\frac{2+\alpha\sqrt{1-x_1}}{\sqrt{2}(\alpha+\sqrt{1-x_1})} - iq_2\ln\frac{\psi(x_1)}{2\sqrt{2}} + q_3\ln\frac{\varphi_+(x_1)}{2\sqrt{2}} - q_4\ln\frac{\varphi_-(x_1)}{2\sqrt{2}} \qquad (4.5)$$

for $-1 < x_1 < x_0$ and

$$-\frac{p}{k\delta} = q_1 \arccos\frac{2+\alpha\sqrt{1-x_1}}{\sqrt{2}(\alpha+\sqrt{1-x_1})} - iq_2\ln\frac{\psi(x_1)}{2\sqrt{2}} + q_3\ln\frac{\sqrt{1-x_0}\,\varphi_+(x_1)}{2\sqrt{2}(\sqrt{2}+\sqrt{1+x_0})} - q_4\ln\frac{\sqrt{1-x_0}\,|\varphi_-(x_1)|}{2\sqrt{2}(\sqrt{2}+\sqrt{1+x_0})} - q_5 \qquad (4.6)$$

for $x_0 < x_1 < 1$.

Here

$$q_1 = s_1, \quad q_2 = s_2, \quad q_3 = \frac{2a_3\varepsilon}{k\sqrt{1+x_0}}, \quad q_4 = \frac{2a_4\varepsilon}{k\sqrt{1+x_0}}$$

$$q_5 = s_3\ln\frac{\sqrt{2}-\sqrt{1+x_0}}{\sqrt{2}+\sqrt{1+x_0}}$$

Equations (4.5) and (4.6) imply that the third form of the solution of the problem, i.e. the formula for the pressure in domain (3) in Fig. 5 is indeed a sum of terms containing a single arc cosine and three hyperbolic arc cosines.

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BIBLIOGRAPHY

1. Lighthill, M. J., Diffraction of blasts, I. Proc. Roy. Soc. Ser. A, Vol. 198, №1055, 1949.
2. Ting, L. and Ludloff, H. F., Aerodynamics of blasts. J. Aeron. Sci. Vol. 19, №5, 1952.
3. Ludloff, H. F., Aerodynamics of blast waves. In collection: Problems of Mechanics. Moscow, Izd. Inostr. Lit., 1955.
4. Arutiunian, G. M., On the interaction of a shock wave and a wedge. Dokl. Akad. Nauk ArmSSR Vol. 46, №4, 1968.

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